Area-slope relation in a simple erosion model

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We discuss area-slope relation of a landform evolved by a simple lattice model of an erosion process. Observing a steady state of the model, where river networks on the surface are stationary, we present the relation between the power exponent of the area-slope relation and the parameters of the erosion process of the model. We show how to determine the parameters of the erosion process from observations of landform by making use of the area-slope relation. This method may be useful for morphological simulations of the landforms.

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Since Mandelbrot's discovery of the concept of fractals [1], it has been recognized that there are many fractals in the nature. One of the typical examples of the fractal structures observed in the nature is a pattern of a river network. Many years ago, even before the discovery of the concept of fractals, the fractality of river networks has been recognized by hydrologists; the self-similarity of river networks was described as Horton's laws [2], and the self-affinity of individual river basins was formulated as Hack's law [3]. In recent years, the fractality of river networks has been considered as a problem of statistical physics, and its power law in the drainage basin area distribution has attracted many physicists' interest. So, physically motivated models of river network formation have been studied to explain the drainage basin area distribution: Sheidegger's river model [4] that was studied by Takayasu *et al.* as a model of a process of aggregation with injection [5], a model based on self-avoiding random walk [6] that was originally proposed by Leopold and Langbein [7], a model of water erosion [8,9], a minimum energy dissipation model [10,11], and so on.

A system with fractality presents some power-law relation of physical quantities in general. In the case of the river networks, area-slope relation is one of the examples. Hack and other investigators pointed out power-law relation between the drainage basin area S and the mean slope δH of individual river basins such as

$$\delta H S^{\alpha} = \text{const} \tag{1}$$

with a scaling exponent α ranging 0.4–0.7 [3,12,13]. Willgoose *et al.* assumed a state of dynamic equilibrium of landform evolution, where the tectonic uplift and the erosion by the water flow on the surface are balance, and explained the relation of Eq. (1) by the effect of erosion [14]. They also showed that their numerical model of landform formation reproduced the relation, Eq. (1). Sinclair and Ball also discussed the relation between the erosion process without effect of tectonic uplift and the area-slope relation [15]. The author and coworkers proposed a simple erosion model of river network formation to reproduce the power-law drainage basin area distribution [9]. In this paper, we discuss the area-slope relation in our erosion model on a land with no tectonic uplift, and present the relation between the parameters in the model and the scaling exponent α. We also discuss a technique to determine the parameters of the erosion process from observations of landform by making use of the area-slope relation.

Our erosion model is defined on a two-dimensional lattice. The model is composed of two elements assigned on each lattice site (x,y): the local height of the land $h(x,y;t)$ and the water flow intensity $s(x,y;t)$. For a given initial landform $h(x,y;0)$ the evolution of the landform is performed by repeating the following procedures. The time is discretized in numerical simulations and the following procedures form a time unit called time step.

(1) Rain fall. For each lattice site (x,y), the water flow intensity $s(x,y;t)$ is increased by a constant s_1.

(2) Water flow. The water flow intensity on the site (x,y), $s(x,y;t)$, is transported to one of its nearest neighbor sites (x',y') that has the lowest value of the local height among the nearest neighbor sites of the site (x,y). The new water flow intensity of the site (x,y), $s(x,y;t+1)$, is determined by the sum of the water flow intensity transported from the nearest neighbor sites.

(3) Water erosion. The height of the site (x,y), $h(x,y;t)$, is decreased by the effect of the transportation of the water flow intensity $s(x,y;t)$. The new height is calculated by

$$h(x,y;t+1) = h(x,y;t) - C\,\delta h(x,y;t)\frac{J(x,y;t)}{1+J(x,y;t)}, \tag{2}$$

where C is a constant and $\delta h(x,y;t)$ is the height difference

$$\delta h(x,y;t) = h(x,y;t) - h(x',y';t), \tag{3}$$

respectively, and J is a water power defined by

$$J(x,y;t) \equiv [\delta h(x,y;t)]^{a-1}[s(x,y;t)]^{b}, \tag{4}$$

where a and b are parameters governing the erosion process.

Namely, our model describes a simplified water erosion process with the effects of spatially and temporally uniform rain fall, conservation of water, uniform geological structure of the land, etc. In the process of the water flow, procedure (2), a lake is formed at the site (x,y) when $h(x,y;t) \leqslant h(x',y';t)$. But, in this paper, the formation of lakes is not important in the discussion. The details of the model are discussed in our previous papers [9]. Numerical simulations in this paper are conducted on a triangular lattice of size

200×200. The system is connected by a periodic boundary condition in x direction. A line of sinks, where the local height is zero and the water is completely drained, is placed at $y=0$. On the other hand, a high wall is placed at $y=200$ to prevent the water in the system from flowing out of the system from the upper edge. All the water in the system is eventually drained at the sink sites. The initial condition of the system $h(x,y;0)$ is set 2000.0 with a slight white noise ranging 0.0–0.1. The parameters are $s_1=10^{-5}$ and $C=0.5$, respectively.

By drawing a link connecting (x,y) and (x',y') for each site, we get complex, branching river networks on the land surface. Each link shows the path of the water transportation on each site. Since the surface of the landform in the model is always exposed to the effect of the erosion, it is natural to expect that the link of a site often changes its direction as the configuration of the height of the neighboring sites changes. However, it was shown that, after sufficiently long evolution time, the system reaches to a kind of steady state where links in the system hardly change their direction. Namely, a river network hardly changes its shape while the surface of the land is under the effect of the erosion. In this steady state, because of the conservation of the water, the water flow intensity $s(x,y;t)$ of a site (x,y) becomes proportional to the drainage basin area of the site (x,y), where the drainage basin area is defined by the number of the upstream links of the site (x,y). We discuss the area-slope relation of the river basins in this steady state.

Since $J/(1+J)$ can be approximated to J when J is small, Eq. (2) with the effect of the tectonic uplift can be rewritten as

$$\frac{\partial h(x,y;t)}{\partial t}=-C[\delta h(x,y;t)]^a[s(x,y;t)]^b. \tag{5}$$

When there is no tectonic uplift, we expect that the height of the landform decreases monotonically. As we mentioned before, the system reaches to a steady state where river networks on the surface hardly change their shapes after long evolution time even under the effect of the erosion process. So, we assume that the landform $h(x,y;t)$ under a steady state is described by

$$h(x,y;t)=f(t)h_0(x,y), \tag{6}$$

where $f(t)$ is a monotonically decreasing function with time and $h_0(x,y)$ is a time-independent, spatial function. Since the river networks on the surface are determined only by the function $h_0(x,y)$, no link changes its direction by the effect of erosion under this assumption. We further assume the form of the function $f(t)$ as

$$f(t)=ct^{-\gamma} \tag{7}$$

with a constant c and a constant exponent γ. Substituting Eq. (6) and Eq. (7) into Eq. (5), we get

$$t^{-\gamma-1}h_0(x,y)=\text{const}\times t^{-a\gamma}[\delta h_0(x,y)]^a[s_0(x,y)]^b, \tag{8}$$

where $\delta h_0(x,y)=h_0(x,y)-h_0(x',y')$ and $s_0(x,y)$ is the drainage basin area that is proportional to the water flow intensity under the steady state, respectively. Since the time-dependent parts of both the sides of Eq. (8) should follow an identical function form, we get

$$\gamma=\frac{1}{a-1} \tag{9}$$

and

$$\delta h_0(x,y)[s_0(x,y)]^{b/a}=\text{const}\times[h_0(x,y)]^{1/a}. \tag{10}$$

Starting from a similar assumption to Eq. (6), Sinclair and Ball derived a similar relation to Eq. (10) and discussed an area-slope relation in a vector form [15].

The relation Eq. (10) is similar to Eq. (1) and the scaling exponent α can be related to the parameters of erosion as

$$\alpha=\frac{b}{a}. \tag{11}$$

But it means that $\delta h_0(x,y)[s_0(x,y)]^{b/a}$ is constant only when it is observed on a contour line of specific height, because $[h_0(x,y)]^{1/a}$ appears in the right side of Eq. (10). This may be considered as a drawback of Eq. (10). However, when we attempt to determine the parameters a and b of the evolution equation by observing an existent landform, Eq. (10) is useful. By observing $\delta h_0(x,y)$ and $s_0(x,y)$ on a specific contour line, we can calculate the scaling exponent $\alpha=b/a$. Once we get the scaling exponent α we can calculate the left side of Eq. (10) at any point on the surface of the land. And it is a power-law function of the height with exponent $1/a$. So, by these two exponents $\alpha=b/a$ and $1/a$, we can get the full set of the parameters of the landform evolution a and b. Though the original area-slope relation, Eq. (1), can determine the scaling exponent α from observations of real topographical data, it cannot fully determine the parameters of landform evolution since the exponent α only gives the ratio of the parameters a and b.

Hereafter, we check the validity of the above discussion by numerical simulations. In the following discussion, we perform numerical simulations for various combinations of the parameters a and b. The ranges of the parameters are $1.5\leqslant a\leqslant3.0$ and $0.5\leqslant b\leqslant2.0$, respectively. The numerical data in the following discussion are, for the most part, the results of the simulations with 10^5 time steps. For systems with most of the combinations of the parameters, the evolution time $t=10^5$ is long enough to reach their steady states. In such cases we can regard $h(x,y;10^5)$ as $h_0(x,y)$ by properly adjusting the coefficient in Eq. (7). The water flow intensity $s(x,y;10^5)$ is also replaced by $s_0(x,y)$. In the cases of the combinations of relatively small a and large b, the system does not reach its steady state within $t=10^5$. Results of such systems are not available because of too long simulation time.

We assumed that a steady state landform decreases its height following a power-law function, Eq. (7). This assumption leads Eq. (9) that means the exponent γ of Eq. (7) de-

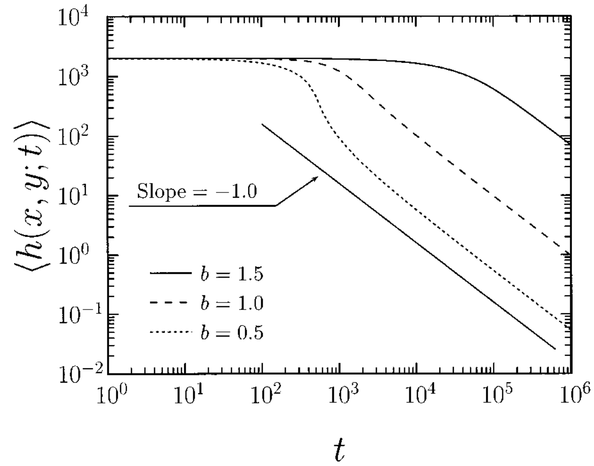

FIG. 1. The time evolution of the average height $\langle h(x,y;t)\rangle$ in the case of $a=2.0$ for various values of b. The straight line shows the slope $\gamma=1$ by Eq. (9). The time and the height in the numerical simulations are dimensionless.

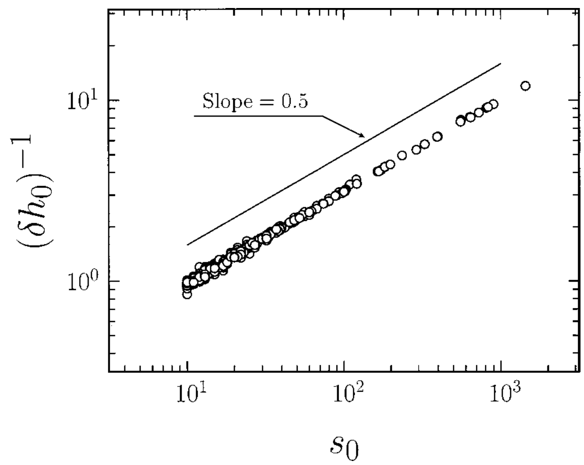

FIG. 3. The plot of $(\delta h_0)^{-1}$ against s_0 in the case of $a=2.0$ and $b=1.0$. The straight line shows the slope $\alpha=1/2$ by Eq. (11). The area and the height in the numerical simulations are dimensionless.

pends only on the parameter a. In Fig. 1 we present the time evolution of average height $\langle h(x,y;t)\rangle$ in log-log scale for the cases of $a=2.0$. The average is taken over all sites in the system. The lines are clear on straight lines with identical slopes. This indicates that the assumption Eq. (7) holds and the slope only depends on the parameter a. The exponent γ of the decay is plotted against the parameter b in Fig. 2. The result shows good agreement with the relation Eq. (9). These results show that our assumption Eq. (7) is reasonably satisfied in the numerical simulations.

Since the assumption in the discussion has been checked, we expect the result of our discussion is valid. First we check the relation between $\delta h_0(x,y)$ and $s_0(x,y)$. We set a threshold height h_t and extract sites (x,y) where $h_0(x,y)\geqslant h_t$ and $h_0(x',y')<h_t$ hold. That is, we extract sites whose links cross the contour line of height h_t. We plot $[\delta h_0(x,y)]^{-1}$ against $s_0(x,y)$ for the extracted sites in log-log scale in Fig. 3. The threshold height is set $h_t=\langle h_0(x,y)\rangle/2$. The points are roughly on a straight line whose slope shows the scaling exponent α. The exponents calculated by the least square method is collectively plotted against b in Fig. 4 with the result of the discussion Eq. (11). The figure proves that the scaling exponent α follows Eq. (11). In Fig. 5 we present $\delta h_0(x,y)[s_0(x,y)]^{\alpha}$ against height in log-log scale for the cases of $a=2.0$. The data show power-law dependence on the height and the exponents of the power law are identical to one another for various b. The slope is very close to the predicted value of $1/a$. So, by calculating the slopes in Fig. 3 and Fig. 5, we can estimate the parameters a and b of the landform evolution.

One of the simplest cases of landform evolution can be described by an equation

$$\frac{\partial h}{\partial t}=c_1-c_2\nabla\cdot\boldsymbol{J}+c_3\nabla^2 h, \tag{12}$$

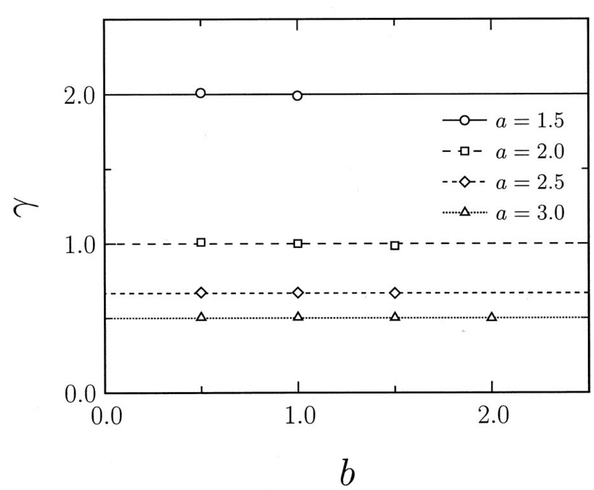

FIG. 2. The exponent γ for various combinations of parameters a and b. The dots show results by numerical simulations, and the lines show the expected values by Eq. (9) for each a.

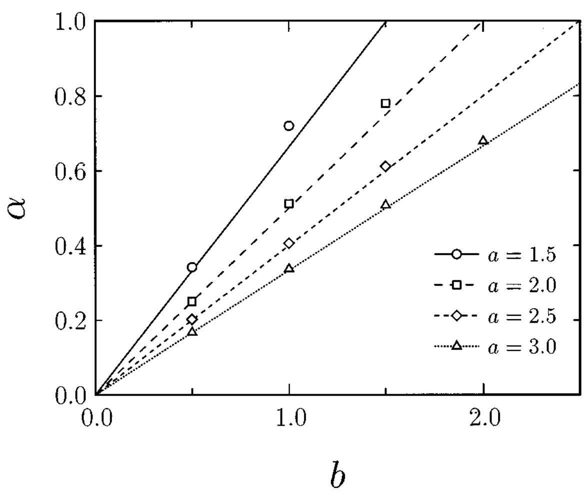

FIG. 4. The exponent α for various combinations of parameters a and b. The dots show results by numerical simulations, and the lines show the values by Eq. (11) for each a.

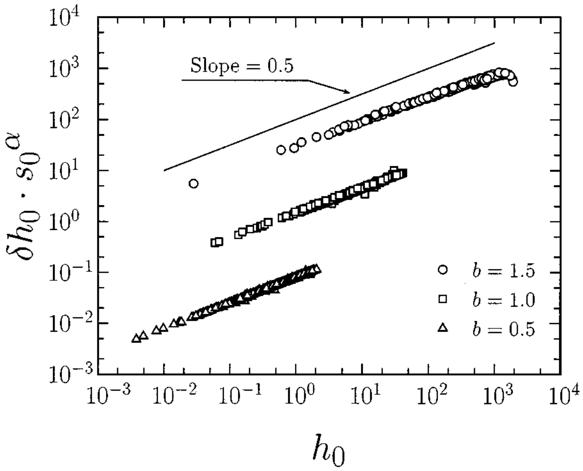

FIG. 5. The plots of $\delta h_0 s_0^\alpha$ against h_0 in the case of $a = 2.0$. The straight line shows the slope $a^{-1} = 1/2$ by Eq. (10). The height, the slope, and the area in numerical simulations are dimensionless.

where h is height, $\boldsymbol{J}$ is sediment flux by water flow, and c_1, c_2, c_3 are constants, respectively [14]. The first term of the right hand side of the equation is the tectonic uplift and the third term means the weathering of the surface of the land. In our model we neglect the effect of the weathering. The second term describes the erosion process by the water flow on the surface. In the case of a lattice model, it is reasonable to construct a model so that the vector of the sediment transport on a lattice site is trained to one of the nearest neighbor sites that has the lowest height among the nearest neighbor sites. It is also reasonable to assume that the sediment in the flow is smoothly washed away without causing sedimentation when the speed of erosion is slow. In this case the divergence of the sediment flux on a site is simply given by the sediment created on the site by water erosion. The creation of sediment flux, $\delta \boldsymbol{J}$, is often assumed to be a power-law function of water flow intensity and local slope [14] such as

$$|\delta \boldsymbol{J}| = \mathrm{const} \times |\nabla h|^a s^b. \tag{13}$$

Under these conditions the erosion process in our model is identical to the discretized version of the erosion process described by Eq. (12).

Our lattice model was originally proposed as a minimal model of erosion to reproduce the statistical properties of river networks. Though the model is seemingly oversimplified, it describes the erosion process described by Eq. (12). With properly chosen parameters, it can reproduce morphological properties of landform formed by erosion process.

The parameters that governs the erosion process affects the resulting landform in its shape, so, it is important to evaluate such parameters from real landforms. By our method presented in this paper, it is possible to determine a set of parameters enough to simulate the landform evolution by analyzing data of an existent landform. By deducing the parameters from an observation, such as a digital elevation map, we may be able to conduct more accurate prediction of the transformation of the landform. The analysis of digital elevation maps in this direction is now under progress.

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